Pin-Ended Tank

Mode shape

$$f(z) = \sin\left(\frac{\pi(z+b)}{l}\right) = \sin\frac{\pi z'}{l}$$

$$\bar{m}_{3,3} = \left[\frac{1}{4\pi^2}\left(\frac{l}{b}\right)^2 + \frac{1}{16}\left(\frac{a}{b}\right)^2\right]\left[1 - \cos\left(\frac{2\pi b}{l}\right)\right]^2 + \sum_{s=1}^{\infty}\frac{2}{\pi^2 s^2} \times \frac{[1+(-1)^{s+1}\cos(2\pi b/l)]^2}{[1-(sl/2b)^2]} \times \left\{1 - \psi_s\left[\frac{(2b/sl)^2 - 2}{(2b/sl)^2 - 1}\right]\right\} + \sum_{s=1}^{\infty}\frac{a/b}{\beta_s(\beta_s^2 - 1)} \times \left[\frac{(\pi a/\beta_s l)^2 + 2}{(\beta_s l/\pi a)^2 + 1}\right]\left\{\coth\left(\frac{2\beta_s b}{a}\right)\left[1 + \cos^2\left(\frac{2\pi b}{l}\right)\right] - 2\cos\left(\frac{2\pi b}{l}\right)\operatorname{csch}\left(\frac{2\beta_s b}{a}\right)\right\} - \sum_{s=1}^{\infty}\frac{1}{2\pi(\beta_s^2-1)} \times \frac{\sin(4\pi b/l)}{(b/l) + (\beta_s b/\pi a)^2(l/a)}$$

$$\bar{m}_{3,s+3} = \frac{(l/b)}{2\pi[1+(\beta_s l/\pi a)^2]}\left\{\frac{1}{\beta_s}\left[2 + \left(\frac{\pi a}{\beta_s l}\right)^2\right] \times \left[\operatorname{csch}\left(\frac{2\beta_s b}{a}\right) - \cos\left(\frac{2\pi b}{l}\right)\coth\left(\frac{2\beta_s b}{a}\right)\right] + \frac{1}{\pi}\left(\frac{l}{a}\right)\sin\left(\frac{2\pi b}{l}\right)\right\}$$

$$\bar{m}_{s+3,s+3} = \frac{a}{4b}\,\frac{(\beta_s^2 - 1)}{\beta_s^3}\coth\left(\frac{2\beta_s b}{a}\right)$$

$$\bar{k}_{3,3} = \frac{g}{16b}\left\{-\frac{8\pi^2 b^2}{l^2} + \left(\frac{\pi^2 a^2}{l^2} + 1\right)\left[\cos\left(\frac{4\pi b}{l}\right) - 1\right]\right\}$$

$$\bar{k}_{3,s+3} = -\frac{\pi g a}{2bl\beta_s^2}\cos\left(\frac{2\pi b}{l}\right)$$

$$\bar{m}_e = \frac{tl\rho_s}{2ba\rho_L}\left[1 + \frac{\pi^2}{2}\left(\frac{a}{l}\right)^2\right]$$

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SEPTEMBER 1963 AIAA JOURNAL VOL. 1, NO. 9

Response of Elastic Columns to Axial Pulse Loading

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In the analysis of columns subjected to axial load pulses of short duration, the nonuniform distribution of longitudinal force along the axis has a significant effect upon the solution. Consequently, it is important to employ a formulation that properly accounts for the propagation of longitudinal and flexural waves and for their interaction. The equations of motion are simultaneous nonlinear partial differential equations. These are solved by an explicit finite difference procedure, taking into account the requirements for stability of the numerical analysis. Solutions are contrasted for two theories, in one of which the effects of rotatory inertia and transverse shear deformation are incorporated, whereas in the other these are omitted. A limited study of the effects of varying system parameters is also presented. For the problems treated, the induced flexural stresses are found to remain small in comparison to the axial stresses.

Introduction

The topic of columns subjected to time-varying axial loads has been considered by many investigators dating back at least to the 1933 paper by Koning and Taub.[1] Much of the literature prior to 1958 has been summarized in a report by Kotowski.[2] For the most part, investigators in this era chose to neglect the effect of longitudinal inertia. This is equivalent to assuming that axial stress waves can be propagated with infinite velocity so that there can be no variation in axial force along the column. For axial forces that vary with a frequency much less than the fundamental frequency of longitudinal vibration, neglect of longitudinal inertia should constitute a satisfactory engineering approximation. Additionally, the paper by Gerard and Becker[3] and the report by Hoppmann[4] should be noted in which the effects of longitudinal waves were considered. However, in both instances the transverse inertia forces were omitted, providing a basis for doubts regarding the physical correspondence of these formulations.

Received April 4, 1963; revision received July 15, 1963.

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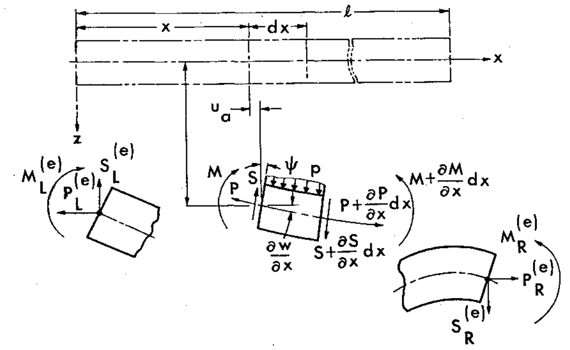

Fig. 1 Element of beam column.

Hoff,[5] in treating the dynamic response of a column in a testing machine, considered the case of a pin-ended column having supports that move together at constant velocity. He also neglected axial inertia but included the axial strain relief provided by lateral displacement. Davidson[6] presented results of calculations for a simply supported column with initial sinusoidal deflection, neglecting axial inertia but retaining the nonlinear strain terms associated with transverse displacement. The load was produced by a mass striking a spring that was attached to one end of the column. For rather low striking velocities, Davidson found satisfactory agreement between his analysis and experiments.

More recently, Sevin[7] has published an extensive analytical study of the response of columns to dynamically applied axial loads in which the effects of longitudinal and transverse inertia plus nonlinear strain terms were included. He specifically treated Hoff's problem and obtained numerical results for four cases on a digital computer. Sevin compared his results with the predictions of Hoff's theory and concluded that axial inertia effects may be of secondary importance insofar as the gross elastic response of conventional structural columns is concerned. However, it should be noted that the results for one of his cases, which featured a high rate of loading, differed markedly from those predicted by Hoff's analysis.

Another analysis of the dynamic response of columns has been reported by Housner and Tso.[8] These authors also neglect longitudinal inertia but include the effects of rotatory inertia and transverse shear deformation. For their assumptions, a modal solution can be obtained and certain conclusions drawn regarding a dominant buckling mode.

Since the present investigation is similar in many respects to Sevin's, the essential differences should be noted. In Sevin's analysis, the axial displacement of one end of the column with respect to the other was specified. This paper treats the case of prescribed axial load, which is believed to be more useful, at least for the applications contemplated. Whereas Sevin treated only the column with simply supported ends, a more general formulation is presented herein with specific attention to the case of columns with free ends. Another distinction is that Sevin did not include the effects of rotatory inertia and transverse shear deformation of column elements in his analysis. Although these effects may be small in comparison with those which were included, they produce a significant change in the mathematical character of the differential equation for the transverse motion of the column. In the usual classification of partial differential equations, Sevin's system of equations is partly parabolic and partly hyperbolic. In order to obtain the proper wave propagation characteristics, which are believed to be essential for analysis of high-speed impact loading, the system should be totally hyperbolic. For this reason, the rotatory inertia and shear deformation terms have been incorporated in the present analysis. The contrast is just that between the Bernoulli-Euler and Timoshenko[9] theories, respectively, for the transverse vibration of beams. The elementary (Bernoulli-Euler) theory has the defect that the velocity of flexural waves becomes unbounded as the wavelength approaches zero. The Timoshenko theory overcomes this difficulty, predicting a finite upper bound for flexural wave velocities which is in good agreement with that for the lowest class of flexural modes from elasticity theory. An assessment of the need for this additional complexity has been made by comparing results of numerical analyses which do and do not include these effects.

Development of Theory

Analytical Formulation

It would be desirable to investigate the possibility of a counterpart of the Euler characteristic load for the buckling of straight columns under dynamic loading. However, the dynamic case appears to possess an infinitude of possible solutions, i.e., one for each stress state of the column at the instant that the column behavior bifurcates from the equally valid straight-column solution. As a consequence, it was decided to present the theory for the dynamic response of a column having an initial out-of-straightness (for which there is a unique solution) and to treat the initially straight column as an interesting limiting case.

In the formulation which follows (hereinafter designated theory I), the effects of longitudinal, transverse, and rotatory inertia are incorporated, as well as the effect of transverse shear deformation. The second-order radial effects associated with longitudinal wave propagation are omitted. The essentials of a corresponding theory that omits rotatory inertia and transverse shear deformation (theory II) are outlined in the Appendix.

Figure 1 illustrates the coordinate geometry and the force and moment resultants that act upon a deformed element of the beam-column. The displacements in the longitudinal and transverse directions may be particularized to

$$u = u_a(x,t) - z\psi(x,t) \tag{1}$$

$$w = w_b(x,t) + w_s(x,t) + w_0(x) \tag{2}$$

In these equations u_a is the axial translation of the centroid of the cross section, and w_b and w_s are the transverse displacements due to bending and to shear deformation, respectively. The initial distribution of transverse displacement is specified by w_0.

These displacement functions are quite restrictive, since they imply that plane cross sections remain plane after loading, merely translating and rotating through the angle ψ rather than warping. This assumption is not compatible with the vanishing of the shearing stresses at the upper and lower surfaces; however, this defect can be remedied when the strain energy of shear deformation is computed. A more general theory, allowing warping of cross sections, has been developed but is not presented, since its additional complexity does not appear to be justified for this investigation.

The strain components pertinent to this analysis are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 = \frac{\partial u_a}{\partial x} - z\frac{\partial \psi}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \tag{3}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \frac{\partial w_0}{\partial x} = \frac{\partial w_s}{\partial x} \tag{4}$$

where use has been made of Eqs. (1) and (2). These strain components have been formulated so that the column is initially unstrained (i.e., for $w = w_0$). Equation (3) for the axial strain contains nonlinear terms that may be of the same order as the linear terms. Their inclusion is necessary in order that the occurrence of a flexural wave may modify the mean longitudinal strain at a cross section.

It is assumed that the column material is homogeneous, isotropic, and linearly elastic. The foregoing expressions for displacements and strains may be employed in formulations of the kinetic and potential energies and of the work done by surface and boundary tractions. These expressions then may be used in conjunction with Hamilton's principle to derive the partial differential equations of motion for a beam-column by the variational method. This procedure leads to the following set of Euler equations.

$$\frac{\partial P}{\partial x} = \rho A \frac{\partial^2 u_a}{\partial t^2} \tag{5}$$

$$\frac{\partial}{\partial x}\left(EI\frac{\partial \psi}{\partial x}\right) + kAG\left(\frac{\partial w}{\partial x} - \frac{\partial w_0}{\partial x} - \psi\right) = \rho I \frac{\partial^2 \psi}{\partial t^2} \tag{6}$$

$$\frac{\partial}{\partial x}\left\{kAG\left(\frac{\partial w}{\partial x} - \frac{\partial w_0}{\partial x} - \psi\right) + P\frac{\partial w}{\partial x}\right\} + p(x,t) = \rho A \frac{\partial^2 w}{\partial t^2} \tag{7}$$

where

$$P = \int_A \sigma_x dA = EA\left\{\frac{\partial u_a}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right\} \tag{8}$$

$$k = \frac{\int_A \gamma_{xz}^2 dA}{A(\partial w_s/\partial x)^2} \tag{9}$$

and

p = distributed transverse load
A = area of cross section
E = Young's modulus
G = shear modulus
I = moment of inertia of cross section about y axis
ρ = mass density of column material

Equations (6) and (7) contain terms that appeared in the coupled equations of the Timoshenko theory employed by Anderson[10] and by Miklowitz[11] plus terms introduced by the axial loading and by the initial displacement. One may use Eq. (8) to eliminate u_a in Eq. (5), thereby obtaining

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho A}\frac{\partial P}{\partial x}\right) + \frac{1}{2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial x}\right)^2 = \frac{1}{EA}\frac{\partial^2 P}{\partial t^2} \tag{10}$$

Equations (6, 7, and 10) comprise a set of three coupled equations for the determination of the three dependent variables w, ψ, and P.

The variational method also provides the mathematical formulation of the "natural" boundary conditions,[12] which may be written as

$$P_{L,R}{}^{(e)} = [P]_{x=0,l} \qquad (u_a \text{ not specified}) \tag{11}$$

$$M_{L,R}{}^{(e)} = [M]_{x=0,l} \qquad (\psi \text{ not specified}) \tag{12}$$

$$S_{L,R}{}^{(e)} = \left[\frac{k}{k'}S + P\frac{\partial w}{\partial x}\right]_{x=0,l} \qquad (w \text{ not specified}) \tag{13}$$

where

$$M = \int_A \sigma_x z dA = -EI\frac{\partial \psi}{\partial x} \tag{14}$$

$$S = \int_A \tau_{xz} dA = k'AG\left(\frac{\partial w}{\partial x} - \frac{\partial w_0}{\partial x} - \psi\right) \tag{15}$$

are the bending moment and the transverse shear force, respectively, and

$$k' = \frac{\int_A \gamma_{xz} dA}{A(\partial w_s/\partial x)} \tag{16}$$

is Timoshenko's shear coefficient.[9] In practice, there will usually occur some simplification of these conditions because of absence of external loading or the specification of geometric quantities.

The mathematical problem expressed by Eqs. (5–13) is truly formidable. The equations of motion are a set of coupled nonlinear partial differential equations. The form of these equations is such that the method of separation of variables is inapplicable. Even if a solution in the form of a series of product terms were valid, it is unlikely that a satisfactory rate of convergence would result for high-speed impact applications where traveling wave behavior may be expected to predominate. Furthermore, it does not appear feasible to apply the method of characteristics to this problem. In the face of these obstacles, the discovery of an exact solution of these equations appears unlikely.

Before considering approximate methods of solution, it is expedient to transform to a set of nondimensional variables that eliminate most of the parametric coefficients from the analysis. By restricting consideration to prismatic columns and introducing the notation

$$\begin{array}{lll} x = r\eta & t = r\tau/c & u_a = r\zeta \\ w = r\xi & P = EAQ & M = EIN/r \\ S = k'AGT & p = EAq/r & c = (E/\rho)^{1/2} \\ r = (I/A)^{1/2} & \beta = kG/E & \lambda = l/r \end{array} \tag{17}$$

into the preceding relations, one obtains a set of differential equations and boundary conditions involving only the shear deformation parameter β. The results of numerical analysis, to be presented subsequently, will be expressed in terms of these dimensionless variables.

Since the differential equations are of second order in time derivatives, initial conditions on each dependent variable and its first time derivative must be specified to complete the definition of a problem. The foregoing formulation (with additional concentrated masses at the boundaries) and a corresponding statement for the axisymmetric response of a thin cylindrical shell were given previously by the author in a company report.[13]

Finite Difference Analysis

The intractable character of the equations of motion led to consideration of approximate procedures for obtaining solutions, specifically the finite difference method and a variational energy method.[13] An explicit finite difference formulation was achieved by use of central difference operators, thereby replacing the continuum problem by a discrete approximation involving nonlinear algebraic equations. The requirements for stability† of solutions of the finite difference equations have been studied in detail and reported elsewhere.[14] Figure 2 depicts typical time variations of the lateral displacement of one end of the column, the only parameters varied being the finite increments $\Delta\eta$, $\Delta\tau$. Curves A, B, and C, corresponding to solutions for which the stability criteria were violated, exhibit rapidly increasing oscillations about the stable solutions for the corresponding $\Delta\eta$ increment size. By contrast, curves D, E, F, and G represent stable solutions that may be seen to be approaching a limit as the increment size is refined. The rate of this convergence can be ascertained by employing a polynomial extrapolation to find $\xi(0, \tau)$ for $\Delta\eta \to 0$. This was done at $\tau = 40$ to obtain the point J and at $\tau = 137.5$ to obtain point K. If these points are regarded as lying on the true solution curve, then the error in curve G is less than 5%. This should be satisfactory for practical purposes, but greater

† This use of the term "stability" should not be confused with the physical problem of the dynamic stability of straight columns which has been investigated and is intended to be the subject of a separate paper.

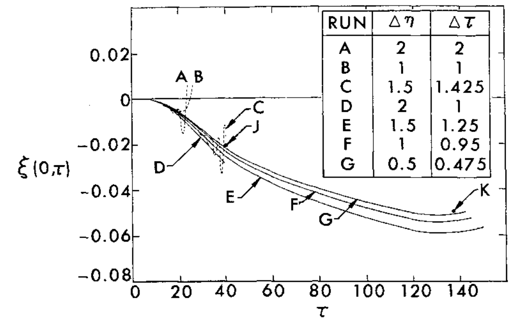

Fig. 2 Comparison of transverse displacements for various increment combinations.

accuracy certainly can be obtained at the expense of using a finer mesh.

The explicit finite difference procedure is well suited to automatic digital computations. Programming for this purpose has been accomplished[15, 16] for both theories I and II.

Numerical Investigation

Although the analytical formulation and the method of solution are applicable to a variety of homogeneous or time-dependent boundary value problems, only the problem of the free-free column subjected to an axial pulse loading applied at one end has been extensively investigated by numerical solution.

Numerical Examples

The boundary conditions to be employed for analysis of the dynamic buckling of a free-ended column are essentially the nondimensional, homogeneous forms of Eqs. (11, 12, and 13), with the exception of an axial load pulse applied at $\eta = 0$. This pulse has an amplitude $\bar{Q}$ and is either a step function or a half-sine pulse of duration $\bar{\tau}$, both commencing at $\tau = 0$. It was assumed that the column was initially at rest (or moving with uniform velocity in the axial direction). To have transverse motion occur when treating this problem by the finite difference method, it is necessary to introduce some perturbation from the straight configuration, such as the initial out-of-straightness $\xi_0 = \bar{\xi} \sin(\alpha\pi\eta/\lambda)$. Alternative methods of achieving lateral response would be to introduce a small transverse disturbance through $q(\eta, \tau)$ or to apply the longitudinal pulse off the centroidal axis (which would be equivalent to a centroidal pulse plus a moment pulse).

The foregoing boundary and initial conditions contain the parameters $\bar{Q}$, $\bar{\tau}$, $\bar{\xi}$, λ, α, β, to which numerical values may be assigned. In studying the effects of parameter variations, it is convenient to define a reference problem for which the parameters have practical magnitudes. The values selected for this standard case are $\bar{Q} = -0.01$, $\bar{\tau} = 40$, $\lambda = 60$, $q = 0$, $\bar{\xi} = 0.25$, $\alpha = 1$, and $\beta = 0.3205$. The considerations involved in this selection merit some discussion. Although the analysis is equally valid for tensile pulses, the principal interest is in buckling. Consequently, negative values are assigned to $\bar{Q}$. The combination of $\bar{Q}$ and λ selected for the standard case corresponds to a peak load that is 3.6 times the Euler (static buckling) load for a pin-ended column of the same slenderness ratio.‡ The standard value of $\bar{\tau}$ was chosen merely to make the pulse duration less than the time required for a longitudinal wave to travel the length of the column. The initial deflection is a half-sine wave with amplitude equal to $\frac{1}{4}$ of the radius of gyration. Although, in principle, the shear parameter β may take on any positive value, practical interest is limited to the range between 0.04 (wood) and 0.4 (cast iron). The value adopted would apply to an aluminum column of solid rectangular cross section. Theory II, in which transverse shear deformation is disregarded, corresponds to $\beta = \infty$.

‡ This corresponds to the second Euler load for a free-ended column, the first Euler load being zero.

Results for the Standard Problem

Certain results from the finite difference solution of the standard problem are presented with a view to characterizing general features of the column response. The time histories of the transverse displacements of the ends of the column are depicted in Fig. 3 for a moderately long run (a range of dimensionless time equal to about 23 times the period required for a longitudinal wave to travel the length of the column). The curves for theory I are based on a computer solution using $\Delta\eta = 0.5$, $\Delta\tau = 0.475$, whereas those for theory II correspond to $\Delta\eta = 1$, $\Delta\tau = 0.5$. These mesh size parameters are sufficiently small to assure adequate convergence of the numerical solutions. It may be seen that there is little difference between the displacements predicted by theories I and II.

In addition to the response to the half-sine pulse, Fig. 3 shows the initial displacements produced by a step-function loading of the same amplitude. These displacements grow rapidly, as is to be expected, since the load continues to act. It should be recalled that the customary small slope linearization ($\partial\xi/\partial\eta \ll 1$) is implicit in the derivation of the equations of motion. For the free-ended column with nonvanishing angular momentum, this restriction will be eventually violated. In the case of the standard problem, this effect does not become significant until times τ of the order of 2000, 77,000 for the step function and the sine pulse, respectively.

The variation of ξ with η (not shown), in addition to the rigid-body rotation component that occurs when the mass center does not lie on the axis of the load pulse, consists of a nonperiodic, small-amplitude oscillation. It would be erroneous to regard the latter as consisting of a superposition of standing wave flexural modes, each having its unique frequency, since the varying axial load would continually modify these quantities.

Were it not for the coupling term $\frac{1}{2}(\partial^2/\partial t^2)(\partial w/\partial x)^2$ in Eq. (10), the axial strains in the prismatic column would be governed by the one-dimensional wave equation; i.e., the initial axial pulse would be propagated along the column without change of form, and reflections at free boundaries would entail a change of sign. Figure 4 depicts the Q-distribution at $\tau = 950$ for theories I and II and also shows the axial strain that would exist if there were no coupling with the lateral response. It may be seen that even after 15 re-

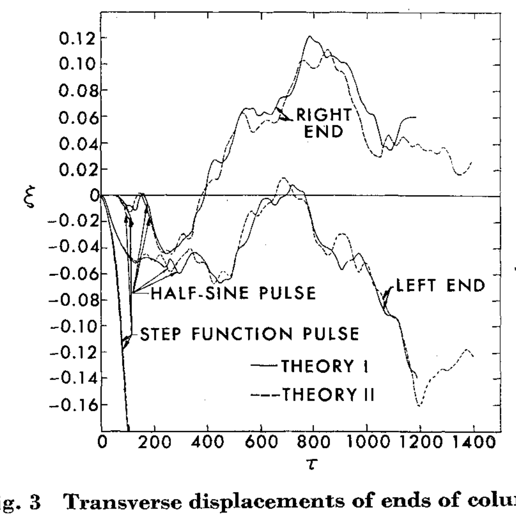

Fig. 3 Transverse displacements of ends of column.

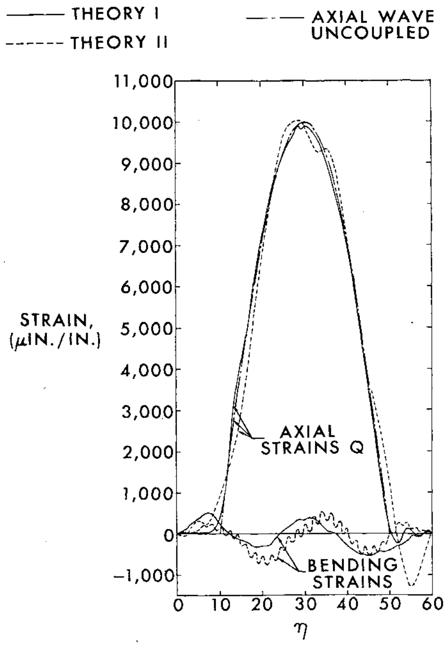

Fig. 4 Distribution of strains at τ = 950.

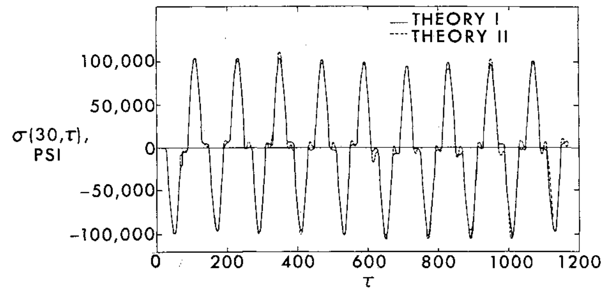

Fig. 5 Variation of total stress at η = 30.

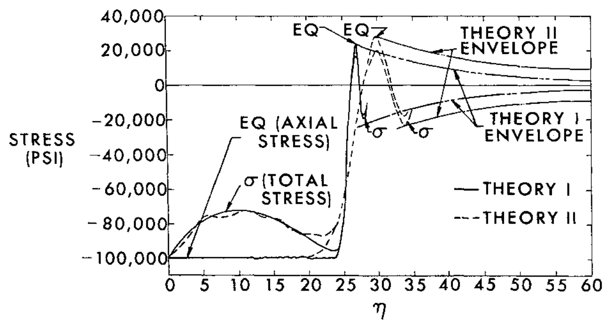

Fig. 6 Stress distribution at τ = 95.

flections there is little distortion of the Q wave. This conclusion is valid throughout the time history of the solution for the standard problem and suggests the possible simplification of neglecting the coupling term in Eq. (10) and introducing the resulting solution for P into Eq. (7). The problem thereby would be reduced to the solution of a pair of linear partial differential equations for ξ and ψ in theory I and a single equation for ξ in the case of theory II. However, the inserted value of axial load would be a variable function of η and τ such that the method of separation of variables is inapplicable and an analytical solution appears unlikely. Clearly, this approximation is less attractive if numerical solution is required.

Also shown in Fig. 4 is the distribution of induced outer fiber§ flexural strains. Although the amplitude of these bending strains is not the absolute maximum that occurred during the solution, it is of the same order of magnitude. Thus, it may be concluded that, for the standard value of $\bar{\xi}$ or less, the bending strains are of secondary importance in comparison to the direct strains produced by the axial pulse. Further evidence of this is provided by Fig. 5, which presents the time history of the total stress (sum of direct and bending stresses) at the midspan of the column. The principal contribution to this almost periodic response is from the multiple reflections of the axial pulse. It is clear that, for the purpose of stress analysis, only a few cycles of the total stress need to be computed. This is particularly true for a real column, for which damping would continually attenuate the response.

A representative distribution of axial and total stresses for the step function loading is displayed in Fig. 6. At this time, the initial compressive wave has been reflected from the right boundary, the unloading wave having progressed slightly more than half the length of the column. The unloading wave is trailing a decaying train of ripples, whereas those that were associated with the initial wave have substantially disappeared even though the system is undamped. In the region to the left of the unloading wave front, there exist induced flexural stresses of magnitude several times greater than those produced by the half-sine pulse. Although the differences between the theory I and theory II solutions are small at this instant, the theory II result does show an excessive distortion of the step-wave front, as should be expected from a diffusion-type theory.

§ The outer fiber distance is taken to be $r\,3^{1/2}$. Where stresses are presented, $E = 10^7$ psi has been employed.

Effects of Parameter Variations

It is not possible to present concisely the results of varying each parameter over its practical range while holding the other parameters constant. However, there are some fairly general conclusions that may be stated on the basis of a limited number of numerical solutions in the neighborhood of the standard problem.

If the amplitude $\bar{Q}$ of the half-sine pulse is varied, holding all other parameters at the values of the standard problem, the effect is substantially what one would expect of a linear system. The lateral displacements and the bending stresses are nearly proportional to $\bar{Q}$ over the range of the latter appropriate for elastic columns.

An increase in the duration $\bar{\tau}$ of the sine pulse causes a somewhat greater than linear increase in displacement peaks; however, these are practically proportional to $\bar{\tau}$ for $0 < \bar{\tau} < \lambda$. There is a gradual increase in the total stresses with $\bar{\tau}$, amounting to 11% between $\bar{\tau} = 10$ and $\bar{\tau} = 120$. For pulses of long duration the response is quasi-static, with the rigid body rotation eventually becoming sufficiently large to violate the assumptions of the theory.

Variation of the amplitude of the initial sinusoidal displacement has shown the lateral displacements and the bending stresses to be proportional to this amplitude for the range $0 < \bar{\xi} < 1$. It was also found that the lateral response uniformly approached zero as $\bar{\xi} \rightarrow 0$, so that no information could be obtained regarding critical loadings for dynamic buckling of straight columns. However, this result justifies the conclusion that an ideally straight column, even though loaded by an axial pulse sufficient to cause physical instability, will not move laterally unless a perturbation is supplied. Of course, this deduction is only of abstract interest, since all physical columns have some imperfections of geometry or loading.

The influence of the parameters α and λ on the solution are directly related to the initial column slope at the loaded end, i.e., to

$$\partial\xi_0/\partial\eta|_{\eta=0} = \alpha\pi\bar{\xi}/\lambda$$

If one increases α, holding this slope constant by reducing ξ, the lateral displacements remain of the same order as for the standard case, but the higher frequency contributions are accentuated, and there is an increase in the flexural stresses. An increase in the slenderness ratio λ, other parameters being held constant, will produce a decrease in the lateral response. This is in direct opposition to the behavior of statically loaded columns, for which an increase in length would result in increased lateral displacement.

The shear deformation parameter β has little effect on the solution. However, the value of β has a marked effect on the convergence of the finite difference analysis. In addition to the stability prerequisites, it is necessary to employ increasingly small $\Delta\eta$, $\Delta\tau$ pairs as β is increased, the requirement being so stringent as to make impractical the use of the theory I digital program to study the limiting solution as $\beta \rightarrow \infty$, i.e., the approach to the theory II solution. Of course, it is unnecessary to be able to accomplish this, since theory II was individually programmed. A more detailed presentation of the effects of parameter variations has been reported.[14]

Concluding Remarks

It may seem surprising that the differences between the solutions for theories I and II are rather small, in view of the known deviations that occur for short pulses in the counterpart beam theories. This result can be traced to the weak coupling between the longitudinal stress waves and the lateral response, which is controlled by the small term $\partial\xi_0/\partial\eta$. For greater values of initial out-of-straightness or loading of a beam-column character including moment or transverse shear pulses at the loaded end or a sufficiently nonuniform lateral load q, greater discrepancies may be expected, with the theory I solution being the more accurate. The difference in computer times for the two theories is not sufficient to justify any compromises in this respect. In fact, as the mesh size is refined, the stability requirements cause theory I to become the more economical of the two.

Appendix

To facilitate comparisons, certain details of the formulation for theory II, in which the effects of rotatory inertia and transverse shear deformation are omitted, should be examined. Starting with the restricted displacements

$$u = u_a(x,t) - z(\partial w_b/\partial x)$$

$$w = w_b(x,t) + w_0(x)$$

one readily finds that Eqs. (5, 8, and 10) for the longitudinal waves apply unchanged but that Eqs. (6) and (7) are replaced by

$$\frac{\partial^2}{\partial x^2}\left\{EI\,\frac{\partial^2(w - w_0)}{\partial x^2}\right\} + \rho A\,\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(P\,\frac{\partial w}{\partial x}\right) = p(x,t)$$

The moment-curvature relationship is

$$M = -EI\,\frac{\partial^2(w - w_0)}{\partial x^2}$$

whereas the shear force[‖] on a cross section normal to the deflected column axis is

$$S = \partial M/\partial x$$

The dimensional form of the natural boundary conditions may be written as

$$P_{L,R}{}^{(e)} = [P]_{x=0,l} \qquad (u_a \text{ not specified})$$

$$M_{L,R}{}^{(e)} = [M]_{x=0,l} \qquad (\partial w/\partial x \text{ not specified})$$

$$S_{L,R}{}^{(e)} = \left[\frac{\partial M}{\partial x} + P\,\frac{\partial w}{\partial x}\right]_{x=0,l} \qquad (w \text{ not specified})$$

The nondimensional quantities defined by Eq. (17) may also be advantageously applied to the formulation for theory II.

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‖ This is the shear force comparable to the S in theory I; it should not be confused with the shear force on a section normal to the x axis, which is frequently used in Euler column analysis.